Role of Charge Separation and Pressure Diffusion in the Gascap of Entry Objects

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Diffusive transport terms, which appear in the hydrodynamic equations that describe the flow over a strongly ablating entry object, are examined. The importance of concentration and thermal diffusion is well established. It is found here that pressure diffusion (which has generally been neglected in favor of concentration diffusion) is of the same order as concentration diffusion in both dissociating and ionizing gases near the shoulder of an entry object. Forced diffusion by electric forces is found to be important. Preliminary experimental evidence of charge separation is cited. It is shown that charge separation does not lead to higher-order terms in the Chapman-Enskog linear flux theory (contrary to an argument in the literature), as long as the usual hydrodynamic formulation is not violated. On these grounds, the Navier-Stokes equations are compatible with a charged gas. The planetary entry flight regime, in which charge separation effects are not negligible, is mapped.

Nomenclature

A,B,C = defined by Eqs. (B8, B9, and B10)
c'_e = defined by Eq. (A11)
c_i = mass fraction of species i
$\bar{c}_p$ = frozen specific heat, $\Sigma_i c_i c_{pi}$
c_{pi} = specific heat at constant pressure for species i
c'_p = elemental mass fraction of element p, $\Sigma_i \alpha_{pi} c_i$
D_{ij} = multicomponent diffusion coefficient of species i in species j
$D_i{}^T$ = thermal diffusion coefficient
$\mathcal{D}_{ij}$ = binary diffusion coefficient
$\mathbf{d}_i$ = macroscopic gradient vector of species i
$\mathbf{E}$ = electrostatic field
e = electron charge
H = total enthalpy, $h + (u^2 + v^2)/2$
h = static enthalpy per unit mass of mixture
j'_e = defined by Eq. (A12)
$\mathbf{j}_i$ = mass flux vector of species i relative to mass averaged velocity
j'_p = defined by Eq. (A9)
j_y = y component of the current
K_i = mass rate of production of species i per unit volume
k = Stefan-Boltzmann constant
L = Avogadro's number
M = molecular weight of mixture, $\Sigma_i x_i M_i$
M_i = molecular weight of species i
M_p = atomic weight of element p
m_i = mass of particle i
n = total number density
n_i = number density of species i
p = pressure
$\mathbf{q}$ = energy flux
R = local longitudinal radius of curvature
Re = Reynolds number $\rho_\infty UR/\mu$
Re' = bulk Reynolds number $\rho_\infty UR/\mu'$
r_b = cross-sectional radius of body, Fig. 4
$\mathbf{r}$ = space coordinate
r' = effective cutoff radius
S = degree of charge separation, Eq. (29)
T = temperature
U = flight speed

u = component of velocity parallel to x coordinate
$\mathbf{V}_i$ = diffusive velocity of species i relative to mass average velocity
v = component of velocity parallel to y coordinate
$\mathbf{X}_i$ = electrostatic force on particle i
x = coordinate parallel to body surface (or Cartesian coordinate in Appendix C)
x_i = mole fraction of species i
Y_i = y component of $\mathbf{X}_i$
y = coordinate normal to body surface (or Cartesian coordinate in Appendix C)
z = Cartesian coordinate
α_{pi} = defined by Eq. (A8)
β_{pi} = number of nuclei of element p in species i
γ_w = net charge density on body surface
δ = shock standoff distance
Δ = charged strata thickness
ϵ = density ratio across shock wave, ρ_∞/ρ_s
η = number of elements in gas mixture
λ = "modified" translational thermal conductivity
Λ = Debye length
μ = viscosity
μ' = bulk viscosity
ν = total number of species
ρ = mass density
ρ_c = net charge density $en \sum_{i=1}^{\nu} \omega_i x_i$
φ = angle in Fig. 4
Ψ = dielectric constant
ω_i = number of electron charges on species i

Subscripts

A = atom
b = body
c = charge
e = electrons
i,j,k = species
I = ion
M = molecule
o = standard atmospheric conditions
p = element p
r = radiative
s = conditions behind shock
x,y = components in those coordinate directions
w = wall conditions
∞ = ambient atmosphere conditions

Superscripts

ν = total number of species in gas mixture

Introduction

DIFFUSION of chemical species enters into the hydro-dynamics of the hot-gas cap over the forward surfaces of objects entering planetary atmospheres in two ways, energy transport and mass transport. Moreover, radiative transfer is influenced by mass transport. Thus, the roles of various kinds of diffusive transport that affect the detailed profiles of composition of radiating and absorbing species throughout the hot gas are assessed.

In a fundamental paper, Butler and Brokaw,[1] in 1957, presented a simplified method of including effects of diffusion of reacting species (in chemical equilibrium and at constant pressure) on energy transport. They expressed the heat flux vector as the product of the "effective thermal conductivity" and the temperature gradient. The thermal conductivity consisted of two parts arising from molecular collisions and diffusion of reacting chemical species. The diffusion of reacting species resulted from their concentration gradients.

The Butler and Brokaw formulation was employed by Hansen[2] to study transport properties of dissociated and ionized air; by Thomas[3] to study carbon dioxide; by Ahtye[4] to study argon (with application to other gases); and by Yos[5] to study nitrogen, oxygen, hydrogen, and air. Although there remain some unresolved conflicts,[6,7] the method led to theoretical predictions of effective thermal conductivity that reasonably agree with electric arc data in nitrogen.[8] It also predicted convective heating rates for planetary entry objects that agree with experimental results in air[9] and carbon dioxide.[10] The method has been particularly useful for flow analyses of the stagnation region[6,10–12] for mixtures of reacting gases whose basic elemental composition is essentially fixed (e.g., for air ~80% nitrogen and 20% oxygen elements).

The crux of the simplification is the imposed condition of no net flux of any atomic species, which in application imposes a condition of invariance of elemental composition. The advantage for the planetary entry application was enormous. It made possible the calculation of transport properties, as a function of pressure and temperature, suitable for insertion into the hydrodynamic equations of flow about the entry object. For stagnation region solutions, where the pressure is almost constant across the flow, the formulation was particularly useful.

However, the flow about strongly ablating entry objects violates the elemental invariance restriction. The elemental composition must vary throughout the flow if the elements that ablate from the surface differ from those of the atmosphere. Strictly speaking, the consequence of applying the restriction is that the stream of ablation vapors originating at the body surface can neither mix nor react with the airstream flowing about the body.‡

Figure 1 illustrates the mixing of ablation products with the hot atmospheric gas in the shock layer. The photographs were obtained by R. B. Pope of NASA Ames Research Center. The nonablating model on the left is geometrically identical to the ablating Teflon model on the right; both photographs were obtained at the same test condition in an argon stream. The exposure on the left was eight times that on the right in order to obtain a photographic image of the nonablating model on the same type film (the bright spot on the model is a reflection). The abundance of light on the right is radiation from ablation products that have diffused,

relative to the mass averaged velocity, into the hot part of the shock layer. So the invariant condition—and the effective thermal conductivity approach—is abandoned.

We are concerned with diffusion processes other than the concentration kind—thermal, pressure, and forced diffusion. Ahtye[4] and Devoto[13] have shown the importance of thermal diffusion to energy transport and have included it in the effective thermal conductivity of argon. Meador and Staton,[14] Devoto,[15] Ahtye,[16,17] and Chung[18] have considered forced diffusion (electric forces) in the absence of pressure gradients. Pressure diffusion has been generally neglected for the planetary entry application.

There is some experimental evidence of the occurrence of charge separation. Unreported experimental results by R. McKenzie of NASA Ames Research Center indicates an accumulation of charge on models in a high-enthalpy gas stream. A difference of 3–5 v was observed between the two ends of the model. Touryan[19] has reported the transfer of charge through a seeded gas flow from one end of a model to the other as a result of thermionic emission from the surface.

Our purpose is to assess the importance of both forced and pressure diffusion by use of the general Chapman-Enskog formulation.[20] We write the governing hydrodynamic equations containing only terms that survive an order of magnitude analysis of the flow about forward facing surfaces of a strongly ablating entry object (Appendix A and Fig. 4). Results of that analysis are used to examine the importance of pressure and forced diffusion on energy and mass flux terms that appear in the hydrodynamic equations.

Analysis

The mass flux vector in a ν component gas mixture

$$\mathbf{j}_i = n_i m_i \mathbf{V}_i = \frac{n^2}{\rho} \sum_{\substack{j=1 \\ j \neq i}}^{\nu} m_i m_j D_{ij} \mathbf{d}_j - D_i^T \frac{\partial \ln T}{\partial \mathbf{r}} \qquad (1)$$

is the mass flux of species i relative to coordinates moving with the mass averaged velocity (Ref. 20, p. 516). In hydrodynamics, it enters the governing equations in two ways: in statements of species continuity [see Eqs. (A5) and (A7) in Appendix A], and in the energy equation [see Eqs. (A4) and (A6) in Appendix A] through the energy flux vector (Ref. 20, pp. 483 and 522—note the radiative energy flux $\mathbf{q}_r$ is treated separately).

The energy flux vector is

$$\mathbf{q} = -\lambda \frac{\partial T}{\partial \mathbf{r}} + \sum_{i=1}^{\nu} h_i \mathbf{j}_i - nkT \sum_{i=1}^{\nu} \frac{1}{n_i m_i} D_i^T \mathbf{d}_i \qquad (2)$$

where the macroscopic gradient vector is (Ref. 20, p. 516)

$$\mathbf{d}_i = \frac{\partial}{\partial \mathbf{r}} \left(\frac{n_i}{n} \right) + \left(\frac{n_i}{n} - \frac{n_i m_i}{\rho} \right) \frac{\partial \ln p}{\partial \mathbf{r}} - \left(\frac{n_i m_i}{p \rho} \right) \left[\frac{\rho}{m_i} \mathbf{X}_i - \sum_{k=1}^{\nu} n_k \mathbf{X}_k \right] \qquad (3)$$

and

$$n = \sum_{i=1}^{\nu} n_i \qquad (4)$$

The $\mathbf{d}_i$ expression contains three kinds of terms: concentration gradient, pressure gradient, and external force. It appears in the mass flux vector multiplied by a multicomponent diffusion coefficient D_{ij}, along with a thermal diffusion term containing the thermal diffusion coefficient D_i^T. Thus, four kinds of diffusive transport (concentration, pressure, forced, and thermal) are represented in the mass flux vector, the diffusive velocity $\mathbf{V}_i$, and the energy flux vector $\mathbf{q}$.

The established role of concentration diffusion and thermal diffusion, Ahtye,[4] in the hydrodynamics of planetary entry

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‡ In some planetary entry applications involving mass addition the condition was employed for energy transport, but was relaxed for mass transport[12] under the assumption that the wall species were essentially like air.

problems has already been mentioned. We now estimate the importance of pressure diffusion and forced diffusion.

Pressure Diffusion

Since the pressure and concentration gradient terms appear together in Eq. (3), we can simply compare the two to evaluate the role of the former. For that purpose, the electrical force terms are neglected in this section.

If we employ the expression

$$\rho = \Sigma_i \, n_i m_i \tag{5}$$

in Eq. (3) we obtain

$$\mathbf{d}_i = \frac{\partial}{\partial r}\left(\frac{n_i}{n}\right) + \frac{n_i}{n}\left[1 - \frac{m_i}{(1/n)\,\Sigma_i \, n_i m_i}\right]\frac{\partial \, \ln p}{\partial r} \tag{6}$$

The order of magnitude analysis of the Navier-Stokes equations applied to the shock layer of a blunted cone indicates that the transverse pressure gradient is such that

$$[(1/p)\partial p/\partial y]_s = 0\,[\tan^2\varphi/\epsilon R] \tag{7}$$

[which can be derived from Eq. (A3) and the oblique shock relations]. In the shoulder region where $\tan^2\varphi \approx 0[1]$, with the assumption $(\delta/R) \approx \epsilon$, Eq. (7) yields

$$[(1/p)\partial p/\partial y]_s = 0[1/\delta] \tag{8}$$

We now make application to illustrative gaseous states behind the shock in the shoulder region.

Dissociating Gas

Here,

$$n = n_A + n_M \tag{9}$$

and

$$m_M = 2m_A \tag{10}$$

For very little dissociation, $n_A/n \ll 1$, and the diffusive velocity for atoms and molecules become, respectively [by use of Eqs. (6) and (1)],

$$\mathbf{V}_A = \frac{2nm_A\mathfrak{D}_{AM}}{\rho}\left[-\frac{n}{n_A}\frac{\partial}{\partial r}\left(\frac{n_A}{n}\right) - \frac{1}{2}\frac{1}{p}\frac{\partial p}{\partial r}\right] \tag{11}$$

and

$$\mathbf{V}_M = \frac{nm_A\mathfrak{D}_{MA}}{\rho}\left[\frac{\partial}{\partial r}\left(\frac{n_A}{n}\right) + \frac{1}{2}\frac{n_A}{n}\frac{1}{p}\frac{\partial p}{\partial r}\right] \tag{12}$$

Behind the shock, we let

$$(\partial/\partial r)(n_A/n) \approx 0[(n_A/n)_s 1/\delta] \tag{13}$$

Use of Eqs. (13) and (8) in (11) and (12) yields

$$\mathbf{V}_A \approx (2nm_A\mathfrak{D}_{AM}/\rho)[-1/\delta - 1/2\delta] \tag{14}$$

and

$$\mathbf{V}_M \approx (n_A m_A\mathfrak{D}_{MA}/\rho)[1/\delta + 1/2\delta] \tag{15}$$

which shows that the diffusive velocity of atoms is much greater than that of molecules (which could also be shown from the definition of the mass averaged velocity) and that the pressure diffusion term $(\partial p/\partial r)/2p$ is the same order as the concentration diffusion term in each expression.

For half-dissociation, $n_A = n_M$, and by similar arguments one can show again that pressure diffusion is the same order as concentration diffusion for the diffusion velocities of both atoms and molecules.

However, for nearly complete dissociation, $n_A \gg n_M$, pressure diffusion can be neglected in favor of concentration diffusion.
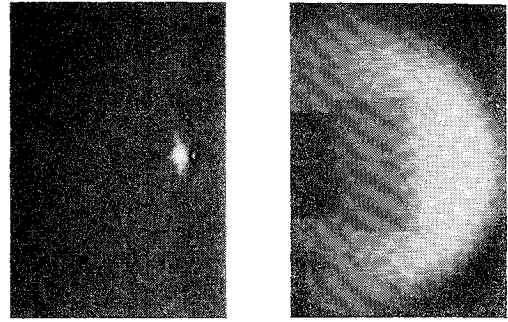

Fig. 1　Nonablating and ablating (Teflon) models in argon stream.

Ionizing Gas

For a mixture of atoms, ions, and electrons,

$$n = n_A + n_I + n_e \tag{16}$$

We write the diffusion velocity of atoms, ions, and electrons from Eq. (1) as

$$\mathbf{V}_A = (n^2/n_A\rho)(m_I D_{AI}\mathbf{d}_I + m_e D_{Ae}\mathbf{d}_e) \tag{17}$$

$$\mathbf{V}_I = (n^2/n_I\rho)(m_e D_{Ie}\mathbf{d}_e + m_A D_{IA}\mathbf{d}_A) \tag{18}$$

$$\mathbf{V}_e = (n^2/n_e\rho)(m_A D_{eA}\mathbf{d}_A + m_I D_{eI}\mathbf{d}_I) \tag{19}$$

For a small degree of ionization, $n_A \gg n_e = n_I$ and $\rho \approx n_A m_A$. Then from Eqs. (3) and (8) we obtain

$$\mathbf{d}_A \approx \frac{\partial x_A}{\partial r} + x_A\left(1 - \frac{n_A m_A}{n_A m_A}\right)\frac{1}{p}\frac{\partial p}{\partial r} = \frac{\partial x_A}{\partial r} \approx \frac{(x_A)_s}{\delta} \tag{20}$$

$$\mathbf{d}_I = \frac{\partial x_I}{\partial r} + x_I\left(1 - \frac{n_A m_I}{n_A m_A}\right)\frac{1}{p}\frac{\partial p}{\partial r} \approx \frac{\partial x_I}{\partial r} \approx \frac{(x_I)_s}{\delta} \tag{21}$$

$$\mathbf{d}_e = \partial x_e/\partial r + x_e(1 - n_A m_e/n_A m_A)(1/p)\partial p/\partial r \approx$$
$$\partial x_e/\partial r + x_e(1/p)\partial p/\partial r \approx (x_e)_s/\delta + (x_e)_s/\delta \tag{22}$$

Thus the pressure diffusion term is important in $\mathbf{d}_e$ alone for small ionization. With that and Ahtye's[4] values for D_{AI}, D_{Ae}, D_{Ie}, and D_{IA} (for argon) used in Eqs. (17) and (18), it can be shown that pressure diffusion is important to the diffusion velocity of atoms and ions, but not to electrons.

Similarly, it can be shown for a high degree of ionization that the pressure gradient term is comparable to the concentration gradient term for $\mathbf{d}_A$ and $\mathbf{d}_I$. Thus, from Eqs. (17–19), pressure diffusion is important for the diffusive velocity of atoms, ions, and electrons.

Forced Diffusion

Here we examine the molecular force terms appearing in the brackets in Eq. (3). The electric field force on a particle is

$$\mathbf{X}_i = \omega_i e\mathbf{E} \tag{23}$$

where ω_i is zero for uncharged particles, -1 for electrons, and equals the number of missing electrons for an ion. The problem is how to express the electric field $\mathbf{E}$.

Ahtye[16,17] discovered that, for a homonuclear mixture of atoms, ions, and electrons, it need not be expressed. One can solve for the macroscopic gradient vectors directly—without separating out the electric field component—if the steady-state condition of Butler and Brokaw[1] is used.

Meador and Staton[14] argued that the electric field must be expressed such that its divergence is zero, i.e., that the hydrodynamic equations are not consistent with a charged gas. We do not agree with their argument for reasons given in Appendix B. Thus, charge separation is allowed in the following.
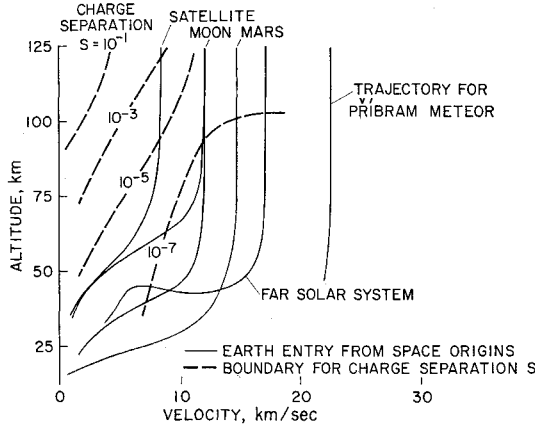
Fig. 2 Trajectories and the forced diffusion regime, $\delta = 0.1$ cm, $\Delta = 0.05\delta$.

Electric Field

We consider the electric field to be primarily a function of the coordinate normal to the surface, and obtain from electrodynamics

$$\partial E_y/\partial y = \rho_c(y)/\Psi \tag{24}$$

The boundary condition is

$$(E_y)_w = \frac{\gamma_w}{\Psi} = -\frac{1}{\Psi} \int_0^\delta \rho_c dy \tag{25}$$

where the second equality is the usual electrostatic condition that makes the field vanish at infinity. Integration of Eq. (24) subject to boundary condition (25) yields

$$E_y(y) = -\frac{1}{\Psi} \int_y^\delta \rho_c dy \tag{26}$$

where

$$\rho_c(y) = e \sum_i \omega_i n_i \tag{27}$$

Equations (26) and (23) give the y component of the electrostatic force. A further approximation of the electrostatic field in terms of the charge density gradient, which avoids the integral of Eq. (26), is presented in Appendix C.

Order of Magnitude Estimate

In the absence of externally applied electric fields, the magnitude of the electrostatic force terms in a partially ionized gas depends on the net charge density variation. For a neutral gas ρ_c is zero, and the force term vanishes. We can estimate the forced diffusion term for various specified degrees of charge separation in a nonneutral gas and compare it with the concentration diffusion term in Eq. (3). For this purpose, we ignore the pressure diffusion term. For the estimate, assume that a thin strata (of thickness Δ) of charged gas (with average charge density ρ_{cs}) exists behind the shock. The field behind this layer is [from Eq. (26)]

$$E_y(\delta - \Delta) = -(\rho_{cs}\Delta/\Psi) = -(\delta/\Psi)(\Delta/\delta)e \sum_i (\omega_i n_i)_s \tag{28}$$

We define the degree of charge separation for a ternary mixture of atoms, ions, and electrons as

$$S = 1 - n_e/n_I = 1/n_I(n_I - n_e) = (1/n_I) \sum_i \omega_i n_i \tag{29}$$

The field becomes

$$E_y(\delta - \Delta) = -(\delta/\Psi)(\Delta/\delta)en_I S_s \tag{30}$$

Estimate

$$(\partial/\partial y)(n_i/n) \approx 0[(x_{i_s} - x_{i_w})/\delta] \tag{31}$$

By use of Eqs. (29–31), the y component of the macroscopic gradient vector for species i [Eq. (3)] can be written

$$d_{i_y}(\delta - \Delta) \approx (x_{i_s} - x_{i_w})/\delta - (e^2/\Psi)[(n_i m_i)_s/p_s \rho_s] \times$$
$$(n_I S - \omega_i \rho/m_i)_s \delta(\Delta/\delta)n_I S_s \tag{32}$$

Forced diffusion can be neglected if the second term on the right is much smaller than the first. Thus the criterion for species i becomes (if $x_{i_w} \ll x_{i_s}$)

$$|(e^2 \delta^2 n_s/\Psi k T_s)[(M_i/M)x_I S - \omega_i]_s x_I(\Delta/\delta)S_s| \ll 1 \tag{33}$$

Consider the quantity in brackets. For increasing ionization (x_I increasing from zero), the criterion is first violated by electrons, followed closely by ions. Generally, the most severe test is for electrons for which Eq. (33) becomes

$$(e^2 \delta^2/\Psi)(n/kT)(\Delta/\delta)x_I S \ll 1 \tag{34}$$

where all quantities are evaluated behind the shock. For a standoff distance, $\delta = 0.1$ cm and strata thickness $\Delta = 0.05 \delta$, the criterion (34) is satisfied above the dashed line for a given S in Fig. 2. That is, for a given charge separation S, forced diffusion is negligible above the line. But it is not negligible below the line, so that for most of the planetary entry regime, the slightest (a hundred thousandth of a percent) charge separation is sufficient to make forced diffusion as important as concentration diffusion in the macroscopic gradient vector for electrons and ions and thus in the mass flux vectors for atoms, ions, and electrons. For larger δ (which is proportional to nose radius), the family of S lines moves up.

Alternatively, instead of setting the charged layer thickness Δ to 0.05 δ, it can be approximated conservatively§ by the Debye thickness Λ such that

$$\Delta = \Lambda = (\Psi k T/e^2 n_I)^{1/2} \tag{35}$$

The criterion (34) becomes

$$(\delta/\Lambda)S_s \ll 1 \tag{36}$$

For $\delta = 0.1$ cm, the criterion is satisfied (and forced diffusion is negligible) above the dashed line in Fig. 3. Forced diffusion is not negligible below the line. By this criterion, one hundredth of a percent charge separation makes forced diffusion important to the entire planetary entry regime. For larger δ, the lines move up as before. For charge layer thickness an order of magnitude larger than the Debye thickness, forced diffusion becomes important for an order of magnitude less charge separation.

The actual extent of charge separation (the local net charge density) can be obtained from the solution of the hydrodynamic equations (Appendix A)—specifically, from the surplus charge diffusion Eq. (A10). We represent the forced diffusion effects in the hydrodynamic equations by use of the electrostatic force expression (23) with the field given by Eq. (26).

Conclusions

The hydrodynamic equations (obtained from an order of magnitude analysis of the equations of change), which describe the flow of a reacting partially ionized gas over the entire forward facing surface of a strongly ablating entry object, are written to include the effects of mass and energy transport by concentration, thermal, pressure, and forced diffusion, as well as the effects of electrostatic body forces.

Although the importance of concentration and thermal diffusion has already been established, that of pressure and forced diffusion has not. The role of the last two is examined

§ The relation is very approximate. For example, Chung[18] showed that the sheath thickness can be an order of magnitude larger than the Debye thickness for a frozen nonneutral boundary layer on a diagnostic probe in a weakly ionized gas.

by an order of magnitude analysis. It is shown that near the windward shoulder of an object entering the atmosphere, a large pressure gradient exists normal to its surface. Consequently, pressure diffusion, which has generally been neglected, is as important as concentration diffusion for 1) atoms and molecules in a slightly dissociated gas, 2) atoms and molecules in a half-dissociated gas, 3) atoms and ions in a slightly ionized gas, and 4) atoms, ions, and electrons in a highly ionized gas.

Forced diffusion (by electrostatic forces) depends on the extent of charge separation in the gas. It is shown (contrary to an argument in the literature) that a charged gas is compatible with linear flux theory. The order of magnitude analysis shows that forced diffusion is as important as concentration diffusion for most of the planetary entry regime if there is even slight charge separation. (The actual extent of charge separation is obtainable from the solution of the set of hydrodynamic equations.) The electrostatic field approximation in terms of the integral of charge density is used in the estimate. A further approximation of the electrostatic field in terms of the local charge density gradient is also presented.

Appendix A: Hydrodynamic Equations

For high-speed flight, thermal radiation can cause the entire gas cap on an entry object to be nonisoenergetic.[11,12] Thus, a thermal boundary layer is not readily defined and it becomes convenient to solve all flow equations from the body to the bow shock without reference to a boundary layer as such. When the equations of change (Ref. 20, p. 698) for the hot gas between the entire body and its bow shock wave are written in orthogonal curvilinear coordinates, the momentum and energy equations each contain approximately 100 terms. To simplify, we have eliminated higher-order terms by an order of magnitude analysis. The body configuration used for this purpose is a nonslender blunted cone (Fig. 4) that typifies a bluff entry object. The individual terms in the equations of change were sized in six regions of the flow, both behind the shock and at the surface: in the stagnation region, on the curved shoulder, and on the cone (afterbody). The basic assumptions for the purpose of size estimates are that the shock layer is thin and that the mass flux normal to the body surface, which includes the surface ablation rate, may be as large as the order of that of the free-stream. Neglecting terms of the order ϵ, Re^{-1}, $(Re')^{-1}$, $(\epsilon Re)^{-1}$,[¶] and $(\epsilon Re')^{-1}$ compared with unity, the resulting
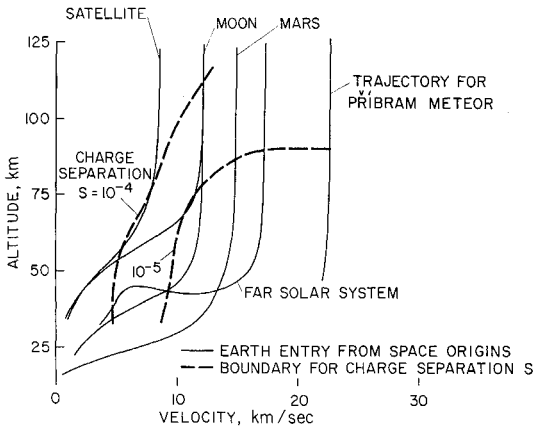

Fig. 3   **Trajectories and the forced diffusion regime,** $\delta = 0.1$ cm, $\Delta = \Lambda$.

[¶] This provides simplification beyond that of Hung Ta Ho and Probstein[21] and Hoshizaki and Wilson[22] and is easily satisfied by $Re \geq 200$.
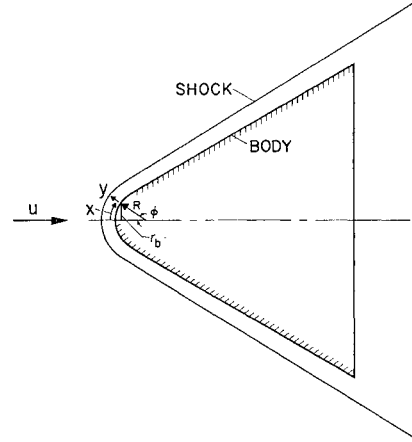

Fig. 4   **Coordinate system.**

hydrodynamic equations are

$$(\partial/\partial x)(\rho u r_b) + (\partial/\partial y)(\rho v r_b) = 0 \qquad \text{(continuity)} \quad (A1)$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{\partial p}{\partial x} \qquad (x \text{ momentum})$$
$$(A2)$$

$$\frac{\partial p}{\partial y} = \frac{\rho u^2}{R} + \sum_{i=1}^{\nu} n_i Y_i \qquad (y \text{ momentum}) \quad (A3)$$

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = -\frac{\partial}{\partial y}(q_y) + \frac{\partial}{\partial y}\left(\mu u \frac{\partial u}{\partial y}\right) - \text{div } \mathbf{q}_r +$$
$$\sum_{i=1}^{\nu} Y_i \frac{j_{iy}}{m_i} \qquad \text{(energy)} \quad (A4)$$

$$\rho u \partial c_i/\partial x + \rho v \partial c_i/\partial y = -(\partial/\partial y)(j_{iy}) + K_i$$
$$\text{(species continuity)} \quad (A5)$$

where the indicated summation is over all species of the gas mixture. These are essentially the boundary-layer equations for a reacting gas, but they apply throughout the flow about the body and are valid for strong ablation. Because ionized species are present at high-speed flight, we have retained force terms arising from electric fields in the y momentum and energy equations.

For chemically reacting gases, it is convenient to replace enthalpy by temperature as a dependent variable. Thus, Eqs. (A2) and (A3) can be combined to yield

$$\rho u \left(\bar{c}_p \frac{\partial T}{\partial x} + \sum_{i=1}^{\nu} h_i \frac{\partial c_i}{\partial x}\right) + \rho v \left(\bar{c}_p \frac{\partial T}{\partial y} + \sum_{i=1}^{\nu} h_i \frac{\partial \bar{c}_i}{\partial y}\right) =$$
$$-\frac{\partial}{\partial y}(q_y) + \mu\left(\frac{\partial u}{\partial y}\right)^2 + u \frac{\partial p}{\partial x} - \text{div } \mathbf{q}_r + \sum_{i=1}^{\nu} Y_i \frac{j_{iy}}{m_i} \quad (A6)$$

which replaces Eq. (A4).

Equation (A5) can be multiplied by α_{pi} and summed over all species i to give the elemental continuity equation for element p

$$\rho u \partial c'_p/dx + \rho v \partial c'_p/\partial y = -(\partial/\partial y)(j'_p) \qquad (A7)$$

where

$$\alpha_{pi} = \beta_{pi} M_p/M_i \qquad (A8)$$

and is zero for electrons, and

$$j'_p = \sum_{i=1}^{\nu} \alpha_{pi} j_{iy} \qquad (A9)$$

Equation (A5) can also be multiplied by $-\omega_i(M_e/M_i)$ and summed over all species i to give the surplus charge diffusion

equation

$$\rho u (\partial c'_e / \partial x) + \rho v (\partial c'_e / \partial y) = -(\partial / \partial y) j'_e \quad \text{(A10)}$$

where

$$c_e' = \sum_{i=1}^{\nu} (-\omega_i) \frac{M_e}{M_i} c_i = \frac{M_e}{M} \sum_{i=1}^{\nu} (-\omega_i x_i) = -\left(\frac{M_e}{e\rho L}\right) \rho_c \quad \text{(A11)}$$

and

$$j'_e = \sum_{i=1}^{\nu} \left(-\omega_i \frac{M_e}{M_i} j_{iy}\right) = -\frac{M_e}{eL} \sum_{i=1}^{\nu} e\omega_i n_i \mathbf{V}_{iy} = -\frac{M_e}{eL} j_y \quad \text{(A12)}$$

Thus c'_e is related to the surplus charge density ρ_c and is zero for a neutral gas, whereas j'_e is related to the y component of the current, j_y. By summing, it can be shown that

$$\sum_{p=1}^{\eta} c'_p + c'_e = 1 \quad \text{(A13)}$$

Equations (A7) and (A10) eliminate the production term since

$$\sum_{i=1}^{\nu} \alpha_{pi} K_i = \sum_{i=1}^{\nu} \omega_i \frac{M_e}{M_i} K_i = 0 \quad \text{(A14)}$$

even for a nonneutral gas mixture not in chemical equilibrium. However, they are especially useful with the chemical equilibrium assumption. Then one can drop the ν species continuity equations, Eq. (A5), corresponding to ν species in favor of η elemental continuity equations, Eq. (A7), corresponding to η elements, plus Eq. (A10) for a nonneutral gas. The solutions of Eqs. (A7) and (A10) give the elemental mass fractions c'_p and surplus charge c'_e, which can be used in a conventional free energy minimization computation of the local species composition.

Appendix B: Compatibility of a Charged Gas with the Hydrodynamic Equations

The argument of Meador and Staton[14] against the compatibility of a charged gas with the hydrodynamic equations is repeated briefly here. It centers around Eq. (3), which combined with Eq. (23), can be written as

$$\mathbf{d}_i = \frac{\partial x_i}{\partial \mathbf{r}} + (x_i - c_i) \frac{\partial \ln p}{\partial \mathbf{r}} - \frac{ec_i}{p} \left(\frac{\rho \omega_i}{m_i} - n \sum_{k=1}^{\nu} \omega_k x_k\right) \mathbf{E} \quad \text{(B1)}$$

The electrodynamic relation

$$\frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{E} = \frac{\rho_c}{\Psi} \equiv en \sum_{k=1}^{\nu} \frac{\omega_k x_k}{\Psi} \quad \text{(B2)}$$

was employed with an assumed relation for the current

$$\mathbf{j} = \sigma_1 \mathbf{E} + \sigma_2 (\partial T / \partial \mathbf{r}) \quad \text{(B3)}$$

The current was set to zero, which resulted in the expression for the field

$$\mathbf{E} = -(\sigma_2 / \sigma_1)(\partial T / \partial \mathbf{r}) \quad \text{(B4)}$$

This expression was used in both equation (B2) and (B1), whereas charge density, effectively

$$en \sum_{k=1}^{\nu} \omega_k x_k$$

from Eq. (B2), was used in (B1), which resulted in $\mathbf{d}_i$ of the form

$$\mathbf{d}_i = \partial x_i / \partial \mathbf{r} + (x_i - c_i) \partial \ln p / \partial \mathbf{r} +$$
$$(c_i \sigma_2 / p \sigma_1)[e\rho \omega_i / m_i + \Psi(\sigma_2 / \sigma_1)\nabla^2 T]\partial T / \partial \mathbf{r} \quad \text{(B5)}$$

The last product $\nabla^2 T(\partial T / \partial \mathbf{r})$ is "third order" in temperature, which is not consistent with linear flux theory. Thus, it was argued, $(\partial / \partial \mathbf{r}) \cdot \mathbf{E}$, and therefore charge density ρ_c must be zero in Eq. (B2) to avoid the third-order expression.

However, that conclusion seems to be a consequence of the use of two relations where one would suffice. There are hydrodynamic equations to calculate every quantity in Eq. (B1) except $\mathbf{E}$. One additional relation for $\mathbf{E}$ [Eq. (B2)] is needed to be solved simultaneously with the hydrodynamic differential equations. Equation (B2) is not (in the hydrodynamic framework) an equation for calculating

$$\sum_{k=1}^{\nu} \omega_k x_k$$

because the mole fractions x_k are calculated from the solutions of the diffusion equations, and ω_k is a set of constants. However, Ref. 14 used Eq. (B2) to calculate

$$\sum_{k=1}^{\nu} \omega_k x_k$$

and employed a second relation [(B3) with $\mathbf{j} = 0$] to calculate $\mathbf{E}$. Of course, both Eqs. (B2) and (B3) are valid and physical, but if both are used in the hydrodynamic framework, (B2) should be regarded as a relation for $\mathbf{E}$, and (B3) a relation for one more unknown, the current $\mathbf{j}$. Clearly, within the hydrodynamic framework, Eq. (B1) is first order without requiring that ρ_c be zero. The electric field $\mathbf{E}$ is simply calculable from the electrodynamic differential Eq. (B2) just as the vectors $(\partial \ln p / \partial \mathbf{r})$ and $(\partial x_i / \partial \mathbf{r})$ are calculable from the hydrodynamic differential equations for insertion into Eq. (B1). Thus, there are no third-order terms and, on these grounds at least, a charged gas is compatible with the hydrodynamic equations.

Alternatively, if it is assumed that there is no current, and the Chapman-Enskog expression for the current

$$\mathbf{j} = e \sum_{i=1}^{\nu} \frac{\omega_i}{m_i} \mathbf{j}_i \quad \text{(B6)}$$

is set to zero, there obtains for the electric field

$$\mathbf{E} = A\partial p / \partial \mathbf{r} + B \Sigma_{ij} \omega_i m_j D_{ij} \partial x_j / \partial \mathbf{r} + C\partial T / \partial \mathbf{r} \quad \text{(B7)}$$

where

$$A = \frac{\Sigma_{ij} \omega_i m_j D_{ij}(x_j - c_j)}{e \Sigma_{ij} \omega_i m_j D_{ij} c_j (\rho \omega_j / m_j - \Sigma_k n_k \omega_k)} \quad \text{(B8)}$$

$$B = p / [e \Sigma_{ij} \omega_i m_j D_{ij} c_j (\rho \omega_j / m_j - \Sigma_k n_k \omega_k)] \quad \text{(B9)}$$

and

$$C = \frac{-(\rho k / n) \Sigma_i \omega_i D_i^T / m_i}{e \Sigma_{ij} \omega_i m_j D_{ij} c_j (\rho \omega_j / m_j - \Sigma_k n_k \omega_k)} \quad \text{(B10)}$$

Substitution of Eqs. (B7) and (23) into (3) yields

$$\mathbf{d}_i = \partial x_i / \partial \mathbf{r} + (x_i - c_i [1 + eA(\{\rho / m_i\}\omega_i -$$
$$n \Sigma_k x_k \omega_k)])(1/p)\partial p / \partial \mathbf{r} - (c_i e / p)(\{\rho / m_i\}\omega_i -$$
$$\Sigma_k n_k \omega_k)(B \Sigma_{k,j} \omega_k m_j D_{kj} \partial x_j / \partial \mathbf{r} + C\partial T / \partial \mathbf{r}) \quad \text{(B11)}$$

Again there are no third-order terms, and the expression is compatible with linear flux theory (Navier-Stokes equations). It is self-consistent with the Chapman-Enskog solution of the Boltzmann equation. Coefficients A, B, and C can be computed by that theory. As before, the mole fractions of all species are obtained from the solutions of the diffusion Eq. (A5) rather than from Eq. (B2). For the zero current approximation, Eq. (B2) is superfluous, and can be omitted.

Appendix C: Electrostatic Field Charge Density Gradient Approximation

The y component of the electrostatic field at the point x, y, z—due to charge $\rho_c dv$ from volume element dv and surface charge $\gamma_w da$ from surface elements da—summed over all gas volume bounded by such surfaces is

$$E_y(x,y,z) = -\frac{1}{4\pi\Psi} \int_{\text{volume}} \frac{\rho_c \cos\theta dv}{r^2} - \frac{1}{4\pi\Psi} \int_{\text{area}} \frac{\gamma_w \cos\theta\, da}{r^2}$$

(C1)

where r is the distance from the point to the element. If the boundary surface lies in the plane $y = 0$, the relation becomes

$$E_y(x,y,z) = -\frac{1}{4\pi\Psi} \left\{ \gamma_w \iint_{\text{area}} \frac{1}{r^2} - \frac{y}{r}\, dx'dz' + \right.$$
$$\left. \iiint_{\text{volume}} \frac{\rho_c(x',y',z')}{r^2} \frac{y'-y}{r}\, dx'dy'dz' \right\} \quad \text{(C2)}$$

The area integral is readily evaluated. The volume integral can be evaluated in an approximate way as follows. Expand $\rho_c(x',y',z')$ as the first two terms of Taylor's series about x, y, z, change to spherical coordinates about x, y, z, and integrate to obtain

$$E_y(x,y,z) = (1/\Psi)[\gamma_w - (r'^2/6)(\partial\rho_c/\partial y)_{x,y,z}] \quad \text{(C3)}$$

From electrostatics, $E_y(\delta) = 0$ which determines the cutoff radius

$$r'^2 = 6\gamma_w/(\partial\rho_c/\partial y)_\delta \approx 6\gamma_w\delta/\rho_c(\delta) \quad \text{(C4)}$$

The right-hand equality is for a linear charge density variation, of course. We have for the y component of the electric field

$$E_y = \frac{\gamma_w}{\Psi}\left[1 - \frac{1}{(\partial\rho_c/\partial y)_\delta} \frac{\partial\rho_c}{\partial y} \right] \quad \text{(C5)}$$

Obviously, the expression should be used with caution or modified to suit other circumstances (such as a chargeless body—$\gamma_w = 0$). We make no attempt at rigor. The object here is to show that the electrostatic field may be thought of as a function of local charge density gradient to a first approximation.

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